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Fixed Point Results in Fuzzy Mappings with Rational Expressions

Abstract

Existence of fixed points for fuzzy mappings on a complete metric space including rational expressions are proved.

Keywords: Fuzzy Metric Space, Common Fixed Point, Complete Metric Space.

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Introduction

In 1965 the fuzzy sets was introduced by Zadeh [23] and measure of fuzzy events in [24]. After that a lot of work has been done regarding fuzzy sets and fuzzy mappings. The concept of fuzzy mappings was first introduced by Heilpern [12], he proved fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem for multi valued mappings of Nadler [17], Vijayaraju and Marudai [21] generalized the Bose and Mukherjee's[2] fixed point theorems for contractive types fuzzy mappings. Marudai and Srinivasan [16] derived the simple proof of Heilpern [12] theorem and generalization of Nadler's [17] theorem for fuzzy mappings.

Review of Literature

Bose and Sahani [3], Butnariu [5],[6],[4], Chang and Huang [9], Chang [8], Chitra [10], Som and Mukharjee [20] studied fixed point theorems for fuzzy mappings. Lee and Cho[14] described a fixed point theorem for contractive type fuzzy mappings which is generalization of Heilpern [12] result. Lee, Cho, Lee and Kim [15] obtained a common fixed point theorem for a sequence of fuzzy mappings satisfying certain conditions, which is generalization of the second theorem of Bose and Sahini [3].

Recently, Rajendran and Balasubramanian [19] worked on fuzzy contraction mappings. More recently in 2007 Vijayaraju and Mohanraj [22] obtained some fixed point theorems for contractive type fuzzy mappings which are generalization of Beg and Azam [1], fuzzy extension of Kirk and Downing [13] and which obtained by the simple proof of Park and Jeong [18] .

Aim of the Study

Although a bulk of research work has been done with different type of fuzzy mappings with different concepts. But in this paper we are going to prove some fixed point and common fixed point theorems in fuzzy mappings in metric space, 2 metric space for a real valued function on a set X and also in 3 metric space.

Preliminaries

To prove the results we need following definitions:

Definition 2.1: (Fuzzy Mappings)

Let X be any metric linear spaces and d be any metric in X. A fuzzy set in X is a function with domain X and values in [0,1]. If A is a fuzzy set and $x \in X$, the function value $A(x)$ is called the grade of membership of x in A. The collection of all fuzzy sets in X is denoted by $\mathcal{F}(x)$.

Let $A \in \mathcal{F}(x)$ and $\alpha \in [0,1]$. The set α –level set of A, denoted by A_α

$A_\alpha = \{x: A(x) \geq \alpha\}$ if $\alpha \in [0,1]$,

Now we distinguish from the collection $\mathcal{F}(x)$ a sub collection of approximate quantities, denoted $W(x)$.

Definition 2.2

A fuzzy subset A of X is an approximate quantity iff its α-level set is a compact subset (non fuzzy) of X for each $\alpha \in [0,1]$, and $\sup A(x) = 1$ $\forall x \in X$.

When $A \in W(x)$ and $A(x_0) = 1$ for some $x_0 \in W(x)$, we will identify A with an approximation of x_0. Then we shall define a distance between two approximate quantities.

Definition 2.3

Let $A, B \in W(x)$, $\alpha \in [0,1]$, define
$p_\alpha(A,B) = \inf d(x,y), \forall\, x \in A_\alpha, B_\alpha$
$D_\alpha(A,B) = dist(A_\alpha, B_\alpha)$,
$d(A,B) = \sup D_\alpha(A,B)$

Where dist. is Hausdorff distance. The function p_α is called α-spaces, and a distance between $A\, and\, B$. It is easy to see that p_α is non decreasing function of α. We shall also define an order of the family $W(x)$, which characterizes accuracy of a given quantity.

Definition 2.4

Let $A, B \in W(x)$. An approximate quantity A is more accurate then B, denoted by $A \subset B, iff\ A(x) \le B(x), for\ each\ x \in X$.

Now we introduce a notion of fuzzy mapping, i.e. a mapping with value in the family of approximate quantities.

Definition 2.5

Let X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping iff F is mapping from the set X into $W(Y), i.e\ F(x) \in W(Y) for\ each\ x \in X$.

A fuzzy mapping F is a fuzzy subset on $X \times Y$ with membership function $F(x,y)$. The function value $F(x,y)$ is grade of membership of $y\ in\ F(x)$.

Let $A \in F(X), B \in F(Y)$ the fuzzy set $F^{-1}(B)\ in\, F(X)$, is defined as

$F^{-1}(B)(x) = \sup\big(F(x,y) \cap B(y)\big)$ where $x \in X, y \in Y$.

First of all we shall give here the basic properties of α-space and α-distance between some approximate quantities.

Lemma 2.1

Let $x \in X$, $A \in W(X)$ and $\{x\}$ be a fuzzy set with membership function equal a characteristic function of set $\{x\}$. If $\{x\}$ is subset of a then $p_\alpha(x,A) = 0$ for each $\alpha \in [0,1]$.

Lemma 2.2

$p_\alpha(x,A) \le d(x,y) + p_\alpha(y,A)\ for\ any\ x, y \in X$.

Lemma 2.3

If $\{x_0\}$ is subset of A, then $p_\alpha(x_0,B) \le D_\alpha(A,B)$ for each $B \in W(X)$.

Lemma 2.4[14]

Let (X,d) be a complete metric linear space, T be a fuzzy mapping from X into $W(X)\ and\ x_0 \in X$, then there exists $x_1 \in X$ such that $\{x_1\} \subset T\{x_0\}$

Lemma 2.5[15]

Let $A, B \in W(X)$ then for each $\{x\} \subset A$, there exists $\{y\} \subset B$ such that $D(\{x\},\{y\}) \le D(A,B)$

Let X be a non empty set and $I = [0,1]$. A fuzzy set of X is an element of I^x. For $A, B \in I^x$ we denote $A \subseteq B$ if and only if $A(x) \le B(x)$ for each $x \in X$.

Definition 2.6[17]

An intuitionist fuzzy set (i-fuzzy set) a of X is an object having the form
$a = \langle A^1, A^2 \rangle$, where $A^1, A^2 \in I^x$ and $A^1(x) + A^2(x) \le 1$ for each $x \in X$.

We denote by $IFS(X)$ the family of all i-fuzzy sets of X.

Definition 2.7[16]

Let x_α be a fuzzy point of X. We will say that $\langle x_\alpha, 1 - x_\alpha \rangle$ is an i-fuzzy point of x and it will be denoted by $[x_\alpha]$. In particular $[x] = \langle \{x\}, 1 - \{x\} \rangle$ will be called an i- point of X.

Definition 2.8[17]

Let $A, B \in IFS(X)$. Then $A \subset B$ if and only if $A^1 \subset B^1$ and $B^2 \subset A^2$

Remark 2;

Notice $[x_\alpha\,] \subset A$ if and only if $x_0 \subset A^1$

Let (X,d) be a metric linear space. The α- level set of A is denoted by

$A_\alpha = \{x : A(x) \ge \alpha\}$ if $\alpha \in]0,1]$ and $A_0 = \{x : A(x) > \alpha\}$, Hadzic [11] called a fuzzy mapping from the set of X into a family $W(X) \subset I^x$ defined as $A \in W(X)$ if and only if A_α is compact and convex in X for each $\alpha \in]0,1]$ and Sup $\{Ax : x \in X\} = 1$.

In this context we give the following definitions.

Definition 2.9[6]

Let X be a metric space and $\alpha \in [0,1]$. consider the following family
$W_\alpha(x) = \{A \in I^x : A_\alpha \text{ is nonempty, compact and convex}\}$

Now we define the family if i-fuzzy sets of X as follows:
$\Phi_\alpha(X) = \{A \in IFS(X) : A^1 \in W_\alpha(x)\}$, it is clear that $\alpha \in I$, $W(X) \subset \Phi_\alpha(X)$.

Definition 2.10[16]

Let x_α be a fuzzy point of X. We will say that is a fixed fuzzy point of the fuzzy mapping F over X if $x_\alpha \subset F(x)$ (i.e. the fixed degree of x is at least α). In particular and according to [4], if $(x) \subset F(x)$, we say that x is a fixed point of F.

Main Results

Theorem 3.1

Let (X, d) is a complete metric space T be fuzzy mapping from X into $W_\alpha(x)$ such that

$$D_\alpha\big(T(x),T(y)\big) \le \alpha^* \frac{\rho_\alpha(x,T(x))\,\rho_\alpha(y,T(x)) + \rho_\alpha(y,T(y))\rho_\alpha(x,T(y))}{\rho_\alpha(x,T(x)) + \rho_\alpha(y,T(y)) + \rho_\alpha(x,T(y)) + \rho_\alpha(y,T(x))}$$
$$+\beta \frac{\rho_\alpha(x,T(x))\rho_\alpha(y,T(y)) + \rho_\alpha(x,T(y))\rho_\alpha(y,T(x)) + [d(x,y)]^2}{\rho_\alpha(x,T(x)) + \rho_\alpha(y,T(y)) + \rho_\alpha(x,T(y)) + \rho_\alpha(y,T(x)) + d(x,y)}$$
$$+\gamma \frac{\rho_\alpha(x,T(x))\rho_\alpha(x,T(y)) + \rho_\alpha(y,T(y))\rho_\alpha(y,T(x)) + [d(x,y)]^2}{\rho_\alpha(x,T(x)) + \rho_\alpha(y,T(y)) + \rho_\alpha(x,T(y)) + \rho_\alpha(y,T(x)) + d(x,y)} + \delta d(x,y)$$

For all $x, y \in X, x \ne y$ with $\rho_\alpha\big(x,T(x)\big) + \rho_\alpha\big(y,T(y)\big) + \rho_\alpha\big(x,T(y)\big) + \rho_\alpha\big(y,T(x)\big) \ne 0$.

And $\alpha^*\beta, \gamma, \delta \in [0,1)$ with $3\alpha + 2\beta + 3\gamma + 3\delta < 3$. then T has fixed point in X.........(3.1.1)

If for some positive integer p, T^p is continuous and then T has a fixed point.

Proof

If there exists $x \in X$ such that x_α is fixed fuzzy point of F, i.e. $x_\alpha \subset T^p(x)$ then $\sum_{n-1}^{\infty} k^n d(x,y) = 0$. Let $x_0 \in K$ and suppose that there exists $x_1 \in \big(T^p(x_0)\big)_\alpha$ such that $\sum_{n-1}^{\infty} k^n d(x_0,x_1) < \infty$. Since $\big(T^p(x_1)\big)_\alpha$ is a nonempty compact subset of X, then there exists $x_2 \subset \big(T^p(x_1)\big)_\alpha$ such that:
$d(x_1,x_2) = p_\alpha\big(t^p(x_1)\big) \le D_\alpha\big(T^p(x_0), T^p(x_1)\big)$

By induction we construct a sequence $\{x_n\}$ in X such that $x_n \subset \big(T^p(x_{n-1})\big)_\alpha$

Therefore its subsequence $\{x_{nk}\}$, where $(n_k = n_p)$ also converges to v. Also,
$T^p(v) = T^p|\lim_{k\to\infty} x_{nk}| = \lim_{k\to\infty} T^p x_{nk} = \lim_{k\to\infty} x_{nk+1} = v$.
So v is fixed point of T^p
Now we show that $Tv = v$. Let m be the smallest positive integer such that $T^m(v) = v$, but $T^n(v) \neq v$, for $n = 1,2,3,- - - - - - - - - - - m-1$.
Therefore we can write
$\rho_\alpha(Tv,v) = D_\alpha(Tv, T(T^{m-1}v))$
If $m > 1$, then from (3.1.1)

$$\leq \alpha^* \frac{\rho_\alpha(v,T(v))D_\alpha(T^{m-1}v,T(v)) + D_\alpha(T^{m-1}v,T(T^{m-1}v))D_\alpha(v,T(T^{m-1}v))}{\rho_\alpha(v,T(v)) + D_\alpha(T^{m-1}v,T(T^{m-1}v)) + \rho_\alpha(v,T(T^{m-1}v)) + D_\alpha(T^{m-1}v,T(v))}$$

$$+\beta \frac{\rho_\alpha(v,T(v))D_\alpha(T^{m-1}v,T(T^{m-1}v)) + \rho_\alpha(v,T(T^{m-1}v))D_\alpha(T^{m-1}v,T(v)) + [\rho_\alpha(v,T^{m-1}v)]^2}{\rho_\alpha(v,T(v)) + D_\alpha(T^{m-1}v,T(T^{m-1}v)) + \rho_\alpha(v,T(T^{m-1}v)) + D_\alpha(T^{m-1}v,T(v))}$$

$$+\gamma \frac{\rho_\alpha(v,T(v))\rho_\alpha(v,T(T^{m-1}v)) + D_\alpha(T^{m-1}v,T(T^{m-1}v))D_\alpha(T^{m-1}v,T(v)) + [\rho_\alpha(v,T^{m-1}v)]^2}{\rho_\alpha(v,T(v)) + D_\alpha(T^{m-1}v,T(T^{m-1}v)) + \rho_\alpha(v,T(T^{m-1}v)) + D_\alpha(T^{m-1}v,T(v)) + \rho_\alpha(v,T^{m-1}v)}$$

$$+\eta[\rho_\alpha(v,T(v)) + D_\alpha(T^{m-1}v, T(T^{m-1}v))] + \mu[\rho_\alpha(v, T(T^{m-1}v)) + D_\alpha(T^{m-1}v, T(v))] + \delta\rho_\alpha(v, T^{m-1}v)$$

$$= \alpha^* \frac{\rho_\alpha(v,T(v))D_\alpha(T^{m-1}v,T(v)) + \rho_\alpha(T^{m-1}v,v)d(v,v)}{\rho_\alpha(v,T(v)) + \rho_\alpha(T^{m-1}v,v) + d(v,v) + d(T^{m-1}v,T(v))}$$

$$+\beta \frac{\rho_\alpha(v,T(v))\rho_\alpha(T^{m-1}v,v) + d(v,v)D_\alpha(T^{m-1}v,T(v)) + [\rho_\alpha(v,T^{m-1}v)]^2}{\rho_\alpha(v,T(v)) + D_\alpha(T^{m-1}v,v) + d(v,v) + D_\alpha(T^{m-1}v,T(v)) + \rho_\alpha(v,T^{m-1}v)}$$

$$+\gamma \frac{\rho_\alpha(v,T(v))d(v,v) + \rho_\alpha(T^{m-1}v,v)D_\alpha(T^{m-1}v,T(v)) + [\rho_\alpha(v,T^{m-1}v)]^2}{\rho_\alpha(v,T(v)) + D_\alpha(T^{m-1}v,T(T^{m-1}v)) + \rho_\alpha(v,T(T^{m-1}v)) + D_\alpha(T^{m-1}v,T(v)) + \rho_\alpha(v,T^{m-1}v)}$$

$$+\eta[\rho_\alpha(v,T(v)) + \rho_\alpha(T^{m-1}v, v)] + \mu[d(v,v) + D_\alpha(T^{m-1}v, T(v))] + \delta\rho_\alpha(v, T^{m-1}v)$$

$$= \alpha^* \frac{\rho_\alpha(v,T(v))[\rho_\alpha(T^{m-1}v,v) + \rho_\alpha(v,T(v))]}{\rho_\alpha(v,T(v)) + \rho_\alpha(v,T^{m-1}v) + D_\alpha(T^{m-1}v,T(v))} + \beta \frac{\rho_\alpha(v,T(v))\rho_\alpha(T^{m-1}v,v) + [\rho_\alpha(v,T^{m-1}v)]^2}{\rho_\alpha(v,T(v)) + D_\alpha(T(v),T^{m-1}v) + 2\rho_\alpha(T^{m-1}v,v)}$$

$$+\gamma \frac{\rho_\alpha(v,T(v))\rho_\alpha(T^{m-1}v,v) + \rho_\alpha(v,T(v)) + [\rho_\alpha(v,T^{m-1}v)]^2}{\rho_\alpha(v,T(v)) + D_\alpha(T(v),T^{m-1}v) + 2\rho_\alpha(T^{m-1}v,v)} + \eta[\rho_\alpha(v,T(v)) + \rho_\alpha(T^{m-1}v,v)]$$

$$+\mu[\rho_\alpha(T^{m-1}v,v) + \rho_\alpha(v,T(v))] + \delta\rho_\alpha(v, T^{m-1}v)$$

$$\leq \alpha^* \frac{\rho_\alpha(v,T(v))[\rho_\alpha(T^{m-1}v,v) + \rho_\alpha(v,T(v))]}{2\rho_\alpha(v,T(v))} + \beta \frac{\rho_\alpha(v,T(v))\rho_\alpha(T^{m-1}v,v) + [\rho_\alpha(v,T^{m-1}v)]^2}{3\rho_\alpha(T^{m-1}v,v)}$$

$$+\gamma \frac{\rho_\alpha(T^{m-1}v,v)[\rho_\alpha(T^{m-1}v,v) + \rho_\alpha(v,T(v))][\rho_\alpha(v,T^{m-1}v)]^2}{3\rho_\alpha(T^{m-1}v,v)} + \eta[\rho_\alpha(v,T(v)) + \rho_\alpha(T^{m-1}v,v)]$$

$$+\mu[\rho_\alpha(T^{m-1}v,v) + \rho_\alpha(v,T(v))] + \delta\rho_\alpha(v, T^{m-1}v)$$

$$\rho_\alpha(Tv,v) \leq \left[\frac{\alpha^*}{2} + \frac{\beta}{2} + \frac{\gamma}{3} + \eta + \mu\right]\rho_\alpha(Tv,v) + \left[\frac{\alpha^*}{2} + \frac{\beta}{2} + \frac{2\gamma}{3} + \eta + \mu\right]\rho_\alpha(T^{m-1}v,v)$$

$$\rho_\alpha(Tv,v) \leq S\rho_\alpha(T^{m-1}v,v), \text{ where } S = \frac{\left[\frac{\alpha^*}{2} + \frac{\beta}{3} + \frac{2\gamma}{3} + \eta + \delta\right]}{1 - \left[\frac{\alpha^*}{2} + \frac{\beta}{3} + \frac{2\gamma}{3} + \eta + \mu\right]} < 1,$$

Because $3\alpha^* + 2\beta + 3\gamma + 6\eta + 6\mu < 3$
On continuing this process, we can write this
$\rho_\alpha(Tv,v) \leq S^m \rho_\alpha(Tv,v)$, which contradicts because $S^m < 1$. so $Tv = v$
That is v is fixed point of T. Uniqueness can be verified also by using simple analysis.

Now we further generalize the result of theorem (3.3.1) which T is neither continuous nor satisfies (3.1.1). In what condition T^m satisfies the same rational expressions and continuous, where m is a positive integer, still T has unique fixed point.

Theorem 3.2: Let T be a self map, defined on a complete metric space X, such that for some positive integer m satisfy the conditions:

$$D_\alpha(T^m(x), T^m(y)) \leq \alpha^* \frac{\rho_\alpha(x,T^m(x))\rho_\alpha(y,T^m(x)) + \rho_\alpha(y,T^m(y))\rho_\alpha(x,T^m(y))}{\rho_\alpha(x,T^m(x)) + \rho_\alpha(y,T^m(y)) + \rho_\alpha(x,T^m(y)) + \rho_\alpha(y,T^m(x))}$$

$$+\beta \frac{\rho_\alpha(x,T^m(x))\rho_\alpha(y,T^m(y)) + \rho_\alpha(x,T^m(y))\rho_\alpha(y,T^m(x)) + [d(x,y)]^2}{\rho_\alpha(x,T^m(x)) + \rho_\alpha(y,T^m(y)) + \rho_\alpha(x,T^m(y)) + \rho_\alpha(y,T^m(x)) + d(x,y)}$$

$$+\gamma \frac{\rho_\alpha(x,T^m(x))\rho_\alpha(x,T^m(y)) + \rho_\alpha(y,T^m(y))\rho_\alpha(y,T^m(x)) + [d(x,y)]^2}{\rho_\alpha(x,T^m(x)) + \rho_\alpha(y,T^m(y)) + \rho_\alpha(x,T^m(y)) + \rho_\alpha(y,T^m(x)) + d(x,y)}$$

$$+\eta[\rho_\alpha(x,T^m(x)) + \rho_\alpha(y,T^m(y))] + \mu[\rho_\alpha(x,T^m(y)) + \rho_\alpha(y,T^m(x))] + \delta d(x,y)$$

For all $x, y \in X, x \neq y$ with

$\rho_\alpha(x, T^m(x)) + \rho_\alpha(y, T^m(y)) + \rho_\alpha(x, T^m(y)) + \rho_\alpha(y, T^m(x)) \neq 0$

$and\ \alpha^*, \beta, \gamma, \delta, \eta, \mu \in [0,1)$ with

$3\alpha^* + 2\beta + 3\gamma + 3\delta + 6\eta + 6\mu < 3$. if T^m is continuous,

Then T has fixed point in X.

Proof: From theorem, we assume that T^m has fixed point v. Also

$$T(v) = T(T^m v) = T^m(Tv) = T^{m+1}(v) = v$$

That is $T(v) = v$. Further since a fixed point of T is also a fixed point of T^m.

Theorem 3.3: Let T be a continuous self mapping defined on a complete 2-Metric space X further T is fuzzy mapping from X into $W_\alpha(X)$ such that

$D_\alpha(T(x), T(y), a) \leq K(M(x, y, a))$,

$D_\alpha(T(x), T(y), a) \leq \alpha^* \frac{\rho_\alpha(x,T(x),a)\rho_\alpha(y,T(x),a)+\rho_\alpha(y,T(y),a)\rho_\alpha(x,T(y),a)}{\rho_\alpha(x,T(x),a)+\rho_\alpha(y,T(x),a)+\rho_\alpha(y,T(y),a)+\rho_\alpha(x,T(y),a)}$

$+\beta \frac{\rho_\alpha(x,T(x),a)\rho_\alpha(y,T(y),a)+\rho_\alpha(x,T(y),a)\rho_\alpha(y,T(x),a)+[d(x,y,a)]^2}{\rho_\alpha(x,T(x),a)+\rho_\alpha(y,T(y),a)+\rho_\alpha(x,T(y),a)+\rho_\alpha(y,T(y),a)+d(x,y,a)}$

$+\gamma \frac{\rho_\alpha(x,T(x),a)\rho_\alpha(x,T(y),a)+\rho_\alpha(y,T(y),a)\rho_\alpha(y,T(x),a)+[d(x,y,a)]^2}{\rho_\alpha(x,T(x),a)+\rho_\alpha(y,T(y),a)+\rho_\alpha(x,T(y),a)+\rho_\alpha(y,T(y),a)+d(x,y,a)}$

$+\delta d(x, y, a)$

For all $\in x, yX,\ x \neq y$ with $\rho_\alpha(x, T(x), a)\rho_\alpha(y, T(y), a) + \rho_\alpha(x, T(y), a)\rho_\alpha(y, T(x), a) \neq 0$.

And $\alpha^*, \beta, \gamma, \delta \in [0,1), a > 0$ with $3\alpha^* + 2\beta + 3\gamma + 3\delta < 3$. K is non decreasing function such that $K: [0, \infty) \to [0, \infty), K(0) = 0$ and $K(t) < t\ \ \forall\, t \in (0, \infty)$, then $\exists\, x \in X$ such that common fuzzy point of T iff $x_0, x_1 \in X$ such that $\sum_{n=1}^{\infty} K^n d(x_0, x_1, a) < \infty$. in particular if $\alpha = 1$, the x is fixed point of T.

Proof: If there exists $x \in X$ such that x_α is fixed fuzzy point of F, i.e. $x_\alpha \subset T(x)$ then $\sum_{n=1}^{\infty} K^n d(x, x, a) = 0$. Let $x_0 \in K$ and suppose that there exists $x_1 \in (T(x_0))_\alpha$ such that

$\sum_{n=1}^{\infty} K^n d(x_0, x_1, a) < \infty$. since $(T(x_0))_\alpha$ is non compact subset of X, then there exists $x_1 \in (F(x_1))_\alpha$, such that

$d(x_1, x_2, a) = \rho_\alpha(x_1, T(x_1), a) \leq D_\alpha(T(x_0), T(x_1), a)$

By induction we construct a sequence $\{x_n\}$ in X such that $x_n \subset (T(x_{n-1}))_\alpha$ and $d(x_n, x_{n-1}, a) \leq D_\alpha(T(x_n), T(x_{n-1}), a)$. since K is given to be non-decreasing, so

$d(x_{n+1}, x_n, a) \leq D_\alpha(T(x_n), T(x_{n-1}), a)$

$\alpha^* \frac{\rho_\alpha(x_n,T(x_n),a)\rho_\alpha(x_{n-1},T(x_n),a)+\rho_\alpha(x_{n-1},T(x_{n-1}),a)\rho_\alpha(x_n,T(x_{n-1}),a)}{\rho_\alpha(x_n,T(x_n),a)+\rho_\alpha(x_{n-1},T(x_{n-1}),a)+\rho_\alpha(x_n,T(x_{n-1}),a)\rho_\alpha(x_{n-1},T(x_n),a)}$

$+\beta \frac{\rho_\alpha(x_n,T(x_n),a)\rho_\alpha(x_{n-1},T(x_{n-1}),a)+\rho_\alpha(x_{n-1},T(x_{n-1}),a)\rho_\alpha(x_{n-1},T(x_n),a)+[d(x_n,x_{n-1},a)]^2}{\rho_\alpha(x_n,T(x_n),a)+\rho_\alpha(x_{n-1},T(x_{n-1}),a)+\rho_\alpha(x_n,T(x_{n-1}),a)\rho_\alpha(x_{n-1},T(x_n),a)+d(x_n,x_{n-1},a)}$

$+\gamma \frac{\rho_\alpha(x_n,T(x_n),a)\rho_\alpha(x_n,T(x_{n-1}),a)+\rho_\alpha(x_{n-1},T(x_{n-1}),a)\rho_\alpha(x_{n-1},T(x_n),a)+[d(x_n,x_{n-1},a)]^2}{\rho_\alpha(x_n,T(x_n),a)+\rho_\alpha(x_{n-1},T(x_{n-1}),a)+\rho_\alpha(x_n,T(x_{n-1}),a)\rho_\alpha(x_{n-1},T(x_n),a)+d(x_n,x_{n-1},a)}$

$+\delta d(x_n, x_{n-1}, a)$

$\leq \alpha^* \frac{d(x_n,x_{n+1},a)[d(x_{n-1},x_n,a)+d(x_n,x_{n+1},a)]}{d(x_n,x_{n+1},a)+d(x_n,x_{n-1},a)+d(x_{n-1},x_{n+1},a)}$

$+\beta \frac{d(x_n,x_{n+1},a)d(x_{n-1},x_n,a)+[d(x_n,x_{n+1},a)]^2}{d(x_n,x_{n+1},a)+d(x_n,x_{n+1},a)+d(x_{n-1},x_{n+1},a)}$

$+\gamma \frac{d(x_{n-1},x_n,a)[d(x_{n-1},x_n,a)+d(x_n,x_{n+1},a)]+[d(x_n,x_{n+1},a)]^2}{2d(x_{n-1},x_{n+1},a)+d(x_{n-1},x_{n+1},a)+d(x_{n+1},x_n,a)}$

$+\delta d(x_n, x_{n-1}, a)$

$\leq \alpha^* \frac{d(x_n,x_{n+1},a)[d(x_{n-1},x_n,a)+d(x_n,x_{n+1},a)]}{2d(x_n,x_{n+1},a)}$

$+\beta \frac{d(x_n,x_{n+1},a)d(x_{n-1},x_n,a)+[d(x_n,x_{n-1},a)]^2}{3d(x_{n-1},x_n,a)}$

$+\gamma \frac{d(x_{n-1},x_n,a)[d(x_{n-1},x_n,a)+d(x_n,x_{n+1},a)]+[d(x_n,x_{n-1},a)]^2}{3d(x_{n-1},x_n,a)}$

$+\delta d(x_n, x_{n-1}, a)$

$= \frac{\alpha^*}{2}[d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)] + \frac{\beta}{3}[d(x_n, x_{n+1}, a) + d(x_n, x_{n-1}, a)]$

$+\frac{\gamma}{3}[2d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)] + \delta d(x_n, x_{n-1}, a)$

$= \left[\frac{\alpha^*}{2} + \frac{\beta}{3} + \frac{2\gamma}{3} + \delta\right] d(x_{n-1}, x_n, a) + \left[\frac{\alpha^*}{2} + \frac{\beta}{3} + \frac{\gamma}{3}\right] d(x_n,\ x_{n+1}, a)$

$\left\{1-\left[\frac{\alpha^*}{2}+\frac{\beta}{3}+\frac{\gamma}{3}\right]\right\} d(x_n,\ x_{n+1},a) \le \left[\frac{\alpha^*}{2}+\frac{\beta}{3}+\frac{2\gamma}{3}+\delta\right] d(x_{n-1},x_n,a)$

$d(x_n,\ x_{n+1},a) \le Kd(x_{n-1},\ x_n,a)$

Where $K = \frac{\left[\frac{\alpha}{2}+\frac{\beta}{3}+\frac{2\gamma}{3}+\delta\right]}{\left\{1-\left[\frac{\alpha}{2}+\frac{\beta}{3}+\frac{\gamma}{3}\right]\right\}} < 1$, because $3\alpha+2\beta+3\gamma+3\delta < 3$

$d(x_n,\ x_{n+1},a) \le Kd(x_{n-1},\ x_n,a)$

… … … … …

… … … … …

$d(x_n,\ x_{n+1},a) \le K^n d(x_0,\ x_n,a)$

By the inequality, we have for $m > n$

$d(x_n,\ x_m,a) \le d(x_n,\ x_{n+1},a) + d(x_{n+1},\ x_{n+2},a) + \cdots \ldots \ldots + d(x_{m-1},\ x_m,a)$

$\le [K^n + \le K^n + \le K^{n+1} + \cdots \ldots \ldots \le K^{m-1}] d(x_0,Tx_0,a)$

$d(x_n,\ x_m,a) \le \frac{K^n}{1-K} d(x_0,Tx_0,a) \to 0, as\ m,n \to \infty$

So $\{x_n\}$ is a Cauchy sequence in X. So by the completeness of X, there is a point $v \in X$ such that $x_n \to v$ as $n \to \infty$

$\rho_\alpha(x,T(x),a) \le d(x,\ x_n,a) + \rho_\alpha(x_n,T(x),a)$

$\le d(x,\ x_n,a) + D_\alpha(T(x_{n-1}),T(x),a) \le d(x,\ x_n,a) + Kd(x_{n-1},x,a)$

Consequently, $\rho_\alpha(x,T(x),a) = 0$, and by **lemma 2.1** $x_\alpha \subset T(x)$

Clearly x_α is a fixed fuzzy point of the fuzzy mapping $T\ over\ X$. In particular if $\alpha = 1$ then x is a fixed point of T.

Now we are proving another fixed point theorem for 2-Metric space.

Theorem 3.4

Let T be a fuzzy mapping on a complete 2-Metric space X. further T satisfies the following conditions:

$$D_\alpha(T(x),T(y),a) \le \alpha^* max \begin{bmatrix} \frac{\rho_\alpha(x,T(x),a)\rho_\alpha(y,T(y),a)}{d(x,y,a)}, \frac{\rho_\alpha(x,T(y),a)\rho_\alpha(y,T(x),a)}{d(x,y,a)} \\ d(x,y,a) \end{bmatrix}$$

For all $x,y \in X, x \ne y$ And $\alpha^*,\beta,\gamma,\delta \in [0,1), a > 0$ with $3\alpha^* + 2\beta + 3\gamma + 3\delta < 3$.

Then T has fixed point in X.

Proof

Let x_0 be an arbitrary point in X and we define a sequence $\{x_n\}$ by means of iterates of T by setting, $T_n(x_0) = x_1$, where n is a positive integer. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T. Taking $x_n \ne x_{n+1}\ for\ all\ n$, then

$d(x_{n+1},\ x_n,a) = d(Tx_n,\ Tx_{n-1},a)$

$$D_\alpha(Tx_n,\ Tx_{n-1},a) \le \alpha\ max \begin{bmatrix} \frac{\rho_\alpha(x_n,T(x_n),a)\rho_\alpha(x_{n-1},T(x_{n-1}),a)}{d(x_n,x_{n-1},a)} \\ \frac{\rho_\alpha(x_n,T(x_{n-1}),a)\rho_\alpha(x_{n-1},T(x_{n-1}),a)}{d(x_n,x_{n-1},a)}, d(x_n,\ x_{n-1},a) \end{bmatrix}$$

$d(x_n,\ x_{n+1},a) \le \alpha\ max[d(x_n,\ x_{n+1},a), d(x_n,\ x_{n-1},a)]$

$d(x_n,\ x_{n+1},a) \le \alpha\ d(x_{n-1},\ x_n,a)$

… … … … …

… … … … …

$d(x_n,\ x_{n+1},a) \le \alpha^n\ d(x_0,\ x_1,a)$

By the triangle inequality, we have for $m > n$

$d(x_n,\ x_m,a) \le\ d(x_n,\ x_{n+1},a) + d(x_{n+1},\ x_{n+2},a) + \cdots \ldots \ldots + d(x_{m-1},\ x_m,a)$

$\le [\alpha^n + \le \alpha^{n+1} + \le \cdots \ldots \ldots + \le \alpha^{m-1}] d(x_0,\ Tx_0,a)$

$d(x_n,\ x_m,a) \le \frac{\alpha^n}{1-\alpha} d(x_0,\ Tx_0,a) \to 0$, as $n \to \infty$

So $\{x_n\}$ is a Cauchy sequence in X. So by the completeness of X, there is a point v∈X such that $x_n \to\ v\ as\ n \to \infty$

But by the continuity of T in X implies

$T(v) = T|\lim_{n\to\infty} x_n| = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = v..$

So v is a fixed point of T in X.

Conclusion

In this manner we have proved some fixed point and common fixed point theorems for fuzzy mappings in complete metric space, 2 metric space for a real valued function on a set X and also in 3 metric space with rational expressions.

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