P: ISSN No. 0976-8602 RNI No.UPENG/2012/42622 VOL.-7, ISSUE-3, July-2018

E: ISSN No. 2349-9443 Asian Resonance

Some Common Fixed Point Results in Fuzzy, Fuzzy 2 and Fuzzy 3-Metric Spaces

Abstract

Some common fixed point results are proved in fuzzy metric, fuzzy 2-metric, fuzzy 3-metric spaces for weak compatible mappings and rational expression.

Keywords: Fuzzy Metric Space, Fuzzy 2-Metric Space, Fuzzy 3-Metric Spaces, Weak Compatible Mappings.

Mathematics Subject Classification

47H10, 54H25.

Rashmi Pathak
Research Scholar,
Deptt.of Mathematics and Computer Science,
U.T.D, Rani Durgavati University,
Jabalpur, M. P., India

Manoj Kumar Shukla
Professor,
Deptt.of Mathematics,
Institute for Excellence in Higher Education,
Bhopal, (M.P.), India

Introduction

More than 52 years back the concept of "Fuzzy Sets" was introduced in 1965, by Zadeh [34]. In continuation Zadeh [35], developed the theory of. "Probability measures of fuzzy events", and produced some important results. Not only results where important it opens a vast windows in the field of fixed point theory with a new approach. After that many authors have expansively developed the theory of fuzzy sets and its applications, specially, Deng [8], Kaleva and Seikkala [20], Kramosil and Michalek [21], have introduced the concept of fuzzy metric spaces in different ways. Many authors [1], [6], [9], [15], [16], [18], [19], [22], [24], [27], [28] have also studied the fixed point theory in the fuzzy metric spaces and [2], [3], [4], [5], [17], [23], [31] have studied for fuzzy mappings which opened a new window for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors. Gahler in a series of papers [11], [12], [13] investigated 2-metric spaces. Sharma, Sharma and Iseki [26] studied for the first time contraction type mappings in 2-metic space. Recently Yadava, Rajpoot and Bhardwaj [32], [33] have also worked on 2-Metric spaces and 2- Banach spaces for rational expressions. Very recently in 2016 Garg, S. and Shukla, M. K. proved some fixed point result in complete fuzzy metric space, fuzzy 2-metric space and fuzzy 3-metric space [29], [30] for weak compatible mappings

Aim of the Study

Our aim in this present paper is to generalize the earlier results of fisher [10] and all others also we extend this result to fuzzy 2 and 3-metric spaces by obtaining some common fixed point theorems on fuzzy metric spaces. Till now work on this approach is not reported.

Preliminaries

We know that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. Before going to our main results the following basic definitions, lemmas and already established results are required:

Definition 2.1 [25]

A binary operation $*: [0,1]\text{x}[0,1]\to[0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian Topological monodies with unit 1 such that $a * b \geq c * d$ whenever $a \geq c \, and \, b \geq d \, for \, all \, a,b,c,d \in [0,1]$

Example of t-norm are $a * b = a\,b$ and $a * b = min\,\{a,b\}$

Definition 2.2 [21]

The $3 - tuple\ (X,M,*)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions for all $x,y,z \in X$ and $s,t > 0$,

(FM-1): $M(x,y,0) = 0$

(FM-2): $M(x,y,t) = 1, \forall\, t > 0 \Leftrightarrow x = y$

(FM-3): $M(x,y,t) = M(y,x,t)$

(FM-4): $M(x,z,t+s) \geq M(x,y,t) * M(z,y,s)$

(FM-5): $M(x, y, a): [0,1) \to [0,1)$ is left continuous

In what follows $(X, M,*)$ will denote a fuzzy metric space.

Note that $M\ (x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M\ (x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞.

Example 2.1 [14]

Let (X, d) be a metric space.

Define

$a * b = a\,b, or\ a * b = min\{a, b\}\ and\ for\ all\ x, y \in X\ and\ t > 0,$

$M(x, y, t) = \frac{t}{t+d(x,y)}$ ... ... (2.2.1.1)

Then $(X, M,*)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 2.3 [15]

Let $(X, M,*)$ is a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$,

$\lim_{n\to\infty} M(x, y, t) = 1$

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1\ \forall\, t > 0$ And $p > 0$

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is called complete.

Let $(X, M,*)$ is a fuzzy metric space with the following condition.

(FM-6): $\lim_{n\to\infty} M(x, y, t) = 1\ \forall\, x, y \in X$

Definition 2.4

A function M is continuous in fuzzy metric space iff whenever

$x_n \to x, y_n \to y \Rightarrow \lim_{n\to\infty} M(x_n, y_n, t) \to M(x, y, t)$

Definition 2.5

Two mappings A and S on fuzzy metric space X are weakly commuting if and only if $M\ (ASu, SAu, t) \geq M\ (Au, Su, t)\ u \in X$

Definition 2.6

A binary operation $* : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monodies with unit 1 such that

$a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$ whenever $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2\ for\ all\ a_1, a_2, b_1, b_2\ and\ c_1, c_2\ are\ in\ [0,1]$.

Definition 2.7

The 3-tuple $(X, M,*)$ is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is continuous t-norm and M is fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions:

(FM-1): $M(x, y, z, 0) = 0$

(FM-2): $M(x, y, z, t) = 1, \forall\, t > 0 \Leftrightarrow x = y$

(FM-3): $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$symmetry about three variable

(FM-4): $M(x, y, z, t_1, t_2, t_3) \geq M(x, y, u, t_1) * M(z, u, z, t_2) * M(u, y, z, t_3)$

(FM-5): $M(x, y, z): [0,1) \to [0,1)$ is left continuous,$x, y, z, u \in X,\ \ t_1, t_2, t_3 > 0$

Definition 2.8

Let $(X, M,*)$ be a fuzzy 2-metric space.

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$,

$\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all $a \in X$ and $t > 0$

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ for all $a \in X$ and $t, p > 0$

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is called complete.

Definition 2.9

A function M is continuous in fuzzy 2-metric space, iff whenever

$x_n \to x, \quad y_n \to y$ Then $\lim_{n\to\infty} M(x_n, y_n, a, t) \to$ Mx, y, a, t for all $a \in X\ t > 0$

Definition 2.10

Two mappings A and S on fuzzy 2-metric space X are weakly commuting iff $M(ASu, SAu, a, t) \geq M\ (Au, Su, a, t)$

Definition 2.11

A binary operation $*: [0, 1]^4 \to [0, 1]$ is called a continuous t-norm if $([0, 1],*)$ is an abelian topological monoid with unit 1 such that ∀ $a_1, a_2, b_1, b_2, c_1, c_2$ and $d_1, d_2 \in [0,1]$ $a_1 * b_1 * c_1 * d_1 \geq a_2 * b_2 * c_2 * d_2$ where $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$ and $d_1 \geq d_2$.

Definition 2.12

The 3-tuple $(X, M,*)$ is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is a continuous t-norm monoid and M is a fuzzy set in $X^4 \times [0, \infty]$ satisfying the following conductions:

$(FM' - 1)$: $M(x, y, z, w, 0) = 0$

$(FM' - 2)$: $M(x, y, z, w, t) = 1, \forall\, t > 0,$

Only when the three simplex $\langle x, y, z, w, \rangle$ degenerate

$(FM' - 3)$: $M(x, y, z, w, t) = M(x, w, z, y, t) = M(z, w, x, y, t) = ---$

$(FM' - 4)$:

$M(x, y, z, w, t_1 + t_2 + t_3) \geq M(x, w, z, u, t_1) * M(x, y, z, w, t_2) * M(x, y, z, w, t_3) * M(x, y, z, w, t_4)$

$(FM' - 5)$: $M(x, y, z, w,): [0,1) \to [0,1]$ is left continuous,

$\forall\, x, y, z, u, w \in X,\ t_1, t_2, t_3, t_4 > 0$

Definition 2.13

Let $(X, M,*)$ be a fuzzy 3-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$, if $\lim_{n\to\infty} M(x_n, x, a, b, t) = 1$ for all $a, b \in X$ and $t > 0$

(2) A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1$ for all $a, b \in X$ and $t, p > 0$

(3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.14

A function M is continuous is fuzzy 3-metric space if

$x_n \to x, y_n \to y$ then $\lim_{n\to\infty} M(x_n, y_n, a, b, t) =$ $Mx, y, a, b, t,\ \forall a, b \in X$ and $T > 0$

Definition 2.15

Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff,

$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t)\ \ \forall\, u, a, b \in X$ and $t > 0$

Definition 2.16

Two self maps f and g of a set X are occasionally weakly compatible $(o.w.c.)$ iff there is a point $x\ in\ X$ which is a coincidence point of f and g at which f and g commute.

Some Basic Results

Lemma 2.1

[15] for all $x, y \in X, M(x, y)$ is non - decreasing.

Lemma 2.2

[7] Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition **(FM -6)** If there exists a number $q \in (0,1)$ such that

$M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t)$, $\forall\ t > 0\ and\ n =$ 1,2,3 ... ..., then $\{y_n\}$ is a Cauchy sequence in X

Lemma 2.3

[24] If for all $x, y \in X, t > 0$ and for a number $q \in (0,1)$,

$M(x, y, qt) \geq M(xc, y, t)$, then $x = y$

Lemma 2.4

Let $(X, M, *)$ be a Fuzzy metric space, A and B are occasionally weakly compatible self maps of X. If $A\ and\ B$ have a unique point of coincidence, $w = Ax = Bx$, then w is the unique common fixed point of $A\ and\ B$.

Proof

Since $A\ and\ B$ are occasionally weakly compatible, there exists a point $x \in X$ such that $Ax = Bx = w$ and $ABx = BAx$. Thus, $AAx = ABx = BAx$, which says that Ax is also a point of coincidence of $A\ and\ B$. Since the point of coincidence $w = Ax$ is unique by hypothesis, $BAx = AAx = Ax$, and $w = Ax$is a common fixed point of $A\ and\ B$. Moreover, if z is any common fixed point of $A\ and\ B$, then $z = Az = Bz = w$ by the uniqueness of the point of coincidence.

Fisher [10] proved the following theorem for three mappings in complete metric space:

Theorem 2.5

Let $S\ and\ T$ be continuous mappings of a complete metric space (X, d) into itself. Then $S\ and\ T$ have a common fixed point in X iff there exists a continuous mapping A of $X\ into\ S\ (X) \cap T\ (X)$, which commutes with $S\ and\ T$ and satisfy

$d(Ax, Ay) \geq \alpha\, d\ (Sx, Ty)\ for\ all\ x, y \in X\ and\ 0 < \alpha < 1.$

Then $S, T\ and\ A$ have a unique common fixed point.

Main Results

Theorem 3.1

Let $(X, M, *)$ be a complete fuzzy metric space with the condition (FM-6) and let $P, Q, F\ and\ G$ be self mappings of X. Let pairs $\{P, F\}\ and\ \{Q, G\}$ be occasionally weakly compatible If there exists $h \in 0, 1)$ such that

(3.1.1)

$$M\big(Px, Qy, (ht)\big) \geq \emptyset\left(\begin{Bmatrix} M(Fx, Gy, t), M(Fx, Px, t), M(Qy, Gy, t), \\ \frac{[M(Px,Gy,t)+M(Fx,Px,t)]}{M(Fx,Px,t)}, \frac{[M(Qy,Fx,t)+M(Qy,Gy,t)]}{M(Qy,Gy,t)} \end{Bmatrix}\right)$$

For all $x, y \in X.\ \phi: [0,1]^5 \rightarrow [0,1]$ such that $\phi(t, 1, 1, t, t) > t\ for\ all\ 0 < t < 1$, then there exists a unique point $w \in X$ such that $Pw = Fw = w$ and a unique point

$X\ such\ that\ Qz = Gz = z. Moreover, z = w,$ so that there is a unique common fixed point of $P, F, Q\ and\ G$.

Proof

Let the pairs $\{P, F\}\ and\ \{Q, G\}$ be occasionally weakly compatible, so there are points $x, y \in X\ such\ that\ Px = Fx\ and\ Qy = Gy.$ We claim that $Px = Qy$. If not, by inequality (3.1.1), we have

$$M\big(Px, Qy, (ht)\big) \geq \emptyset\left(\begin{Bmatrix} M(Fx, Gy, t), M(Fx, Px, t), M(Qy, Gy, t), \\ \frac{[M(Px,Gy,t)+M(Fx,Px,t)]}{M(Fx,Px,t)}, \frac{[M(Qy,Fx,t)+M(Qy,Gy,t)]}{M(Qy,Gy,t)} \end{Bmatrix}\right)$$

$$= \emptyset\left(\begin{Bmatrix} M(Fx, Gy, t), M(Fx, Px, t), M(Qy, Gy, t), \\ \frac{M(Px,Qy,t)+M(Px,Px,t)}{M(Px,Px,t)}, \frac{M(Qy,Px,t)+M(Qy,Qy,t)}{M(Qy,Qy,t)} \end{Bmatrix}\right)$$

$$\emptyset\big(M(Px, Qy, t), 1, 1, M(Px, Qy, t), M(Px, Qy, t)\big) > M(Px, Qy, t)$$

This is a contradiction, therefore $Px = Qy, i.e. Px = Fx = Qy = Gy$. Suppose that there is another point z such that $pz = fz$ then by (3.1.1) we have $Pz = Fz = Qy = Gy,$ so $Px = Pz\ and\ w = Px = Fx$ is the unique point of coincidence of $P\ and\ F$. By Lemma (2.4) w is the only common fixed point of $P\ and\ F.$ Similarly there is a unique point $z \in X\ such\ that\ z = Qz = Gz.\ Assume\ that\ w \neq z.$ We have

$$M\ (w, z, ht) = M\ (Pw, Fz, ht)$$
$$\geq \emptyset\left(\begin{Bmatrix} M(Fw, Gz, t), M(Fw, Pw, t), M(Qz, Gz, t), \\ \frac{[M(Pw,Gz,t)+M(Fw,Pw,t)]}{M(Fw,Pw,t)}, \frac{[M(Qz,Fw,t)+M(Qz,Gz,t)]}{M(Qz,Gz,t)} \end{Bmatrix}\right)$$
$$\geq \emptyset\left(\begin{Bmatrix} M(w, z, t), M(w, w, t), M(z, z, t), \\ \frac{[M(w,z,t)+M(w,w,t)]}{M(w,w,t)}, \frac{[M(z,w,t)+M(z,z,t)]}{M(z,z,t)} \end{Bmatrix}\right)$$
$$= \emptyset\big(M(w, z, t)\big) = M(w, z, t)$$

Therefore we have $z = w$.

Hence z is a common fixed point of $P, F, Q\ and\ G$. The uniqueness of the fixed point holds from (3.1.1).

Theorem 3.2

Let $(X, M, *)$ be a complete fuzzy 2-metric space and let $S^r\ and\ T^r$ be continuous mappings of $X\ in\ X$, then $S^r\ and\ T^r$ have a common fixed point in X if there exists continuous mapping $A^r\ of\ X$ into $S^r\ (X) \cap T^r\ (X)$ which weakly compatible with $S^r\ and\ T^r$ and

$$M(A^r x, A^r y, q^r t, a) \geq min\begin{Bmatrix} M(A^r y, A^r y, t, a), \frac{M(S^r x, A^r x, t, a)}{M(A^r y, A^r y, a, t)}, \frac{M(S^r x, T^r y, a, t)}{M(S^r x, A^r y, a, t)}, \\ M(A^r x, T^r y, a, t), M(S^r x, A^r y, a, t) \end{Bmatrix}$$

…..(3.2.1)

for all $x, y, a \in X, t > 0, and\ 0 < q < 1.$

And $\lim_{n\to\infty} M(x, y, z, a, t) = 1,\ \forall\ x, y, z, a \in X$............(3.2.2)

Then S^r, T^r and A^r have a unique common fixed point.

Proof

We define a sequence $\{x_n\}$ such that $A^r x_{2n} = S^r x_{2n-1}$
and $S^r x_{2n-1} = T^r x_{2n}, n = 1,2, \dots\dots$
We shall prove that $\{A^r x_{2n}\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = A^r x_{2n+1}$ in (3.2.1), we write

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq \min\left\{\begin{matrix} M(A^r x_{2n+1}, A^r x_{2n+1}, a, t), \frac{M(S^r x_{2n}, A^r x_{2n}, a, t)}{M(A^r x_{2n+1}, A^r x_{2n+1}, a, t)}, \\ \frac{M(S^r x_{2n}, T^r x_{2n+1}, a, t)}{M(S^r x_{2n}, A^r x_{2n+1}, a, t)}, M(A^r x_{2n}, T^r x_{2n+1}, a, t) \\ M(S^r x_{2n}, A^r x_{2n+1}, a, t) \end{matrix}\right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq \min\left\{\begin{matrix} M(A^r x_{2n+1}, A^r x_{2n+1}, A^r, t), \frac{M(S^r x_{2n+1}, A^r x_{2n}, a, t)}{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}, \\ \frac{M(S^r x_{2n+1}, A^r x_{2n}, a, t)}{M(A^r x_{2n}, A^r x_{2n+1}, a, t)}, M(A^r x_{2n}, T^r x_{2n}, a, t) \\ M(A^r x_{2n+1}, A^r x_{2n+1}, a, t) \end{matrix}\right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq M(A^r x_{2n+1}, A^r x_{2n}, a, q^r t) =$$

$$\min\left\{M(A^r x_{2n}, A^r x_{2n+1}, a, t), \frac{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}, \frac{M(A^r x_{2n+1}, A^r x_{2n}, a, t)}{1}, 1, 1\right\} \geq$$

$\min\{M(A^r x_{2n-1}, A^r x_{2n}, a, t/$
qr,M(Arx2n,Arx2n−1,a,t/qr)M(Arx2n+1,Arx2n,a,t/qr),M(Arx2n,Arx2n−1,a,t/qr)1,1,1

Therefore

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq M(A^r x_{2n}, A^r x_{2n+1}, a, t/q^r)$$

By induction

$$M(A^r x_{2k}, A^r x_{2m+1}, a, q^r t) \geq M(A^r x_{2m}, A^r x_{2k+1}, a, t/q^r)$$

For every k and m in N, Further if $2m + 1 > 2k$, then

$$M(A^r x_{2k}, A^r x_{2m+1}, a, q^r t) \geq M\left(A^r x_{2m}, A^r x_{2k+1}, a, \frac{t}{q^r}\right) \geq \cdots \dots\dots\dots\dots$$
$$\dots \geq M\left(A^r x_0, A^r x_{2m+1}, a, \frac{t}{q^{2kr}}\right) \geq \cdots \dots\dots\dots \quad (3.2.3)$$

If $2k > 2m + 1$, then

$$M(A^r x_{2k}, A^r x_{2m+1}, a, q^r t) \geq M\left(A^r x_{2k+1}, A^r x_{2m}, a, \frac{t}{q^r}\right) \geq \cdots \dots\dots$$
$$\geq M\left(A^r x_{2k-(2m-1)}, A^r x_0, a, \frac{t}{q^{(2m-1)r}}\right) \dots\dots\dots\dots\dots \quad (3.2.4)$$

By simple induction with (3.2.3) and (3.2.4) we have

$$M(A^r x_n, A^r x_{n+p}, q^r t) \geq M(A^r x_0, A^r x_p, t/q^{nr})$$

For $n = 2k, p = 2m + 1$ or $n = 2k + 1, p = 2m + 1$ and by (FM-4)

$$M(A^r x_n, A^r x_{n+p}, a, q^r t) \geq \min\left\{M\left(A^r x_0, A^r x_1, a, \frac{t}{2q^{nr}}\right), M\left(A^r x_1, A^r x_p, a, \frac{t}{q^{nr}}\right)\right\} \dots\dots\dots \quad (3.2.5)$$

If $n = 2k, p = 2m$ or $n = 2k + 1, p = 2m$
For every positive integer p and n in N, therefore

$$\left(A^r x_0, A^r x_p, a, \frac{t}{q^{nr}}\right) \to 1 \text{ as } n \to \infty$$

Thus $\{A^r x_n\}$ is a Cauchy sequence. Since the space X is complete there exists $z \in X$, such that

$$\lim_{n\to\infty} A^r x_n = \lim_{n\to\infty} S^r x_{2n+1} = \lim_{n\to\infty} T^r x_{2n} = z$$

It follows that $A^r z = S^r z = T^r z$ and therefore by weak compatibility

$$M(A^r z, A^{2r} z, a, q^r t) \geq \min\left\{\begin{matrix} M(T^r A^r z, A^r, A^r z, a, t), \frac{M(S^r z, A^r z, a, t)}{M(A^r A^r z, T^r A^r z, a, t)}, \frac{M(S^r z, T^r A^r z, a, t)}{M(S^r z, A^r A^r z, a, t)}, \\ M(A^r z, T^r A^r z, a, t), M(S^r z, A^r A^r z, a, t) \end{matrix}\right\}$$

$$M(A^r z, A^{2r} z, a, q^r t) \geq M(A^r z, T^r A^r z, a, t) \geq M(S^r z, A^r T^r z, a, t) \geq M(A^r z, A^{2r} z, a, t) \dots\dots\dots \geq M\left(A^r z, A^{2r} z, a, \frac{t}{q^{nr}}\right)$$

$$\text{since } \lim_{n\to\infty} M\left(A^r z, A^{2r} z, a, \frac{t}{q^{nr}}\right) = 1 \Longrightarrow A^r z = A^{2r} z$$

Thus z is common fixed point of A, S and T.

For **uniqueness**, let $w\ (w \neq z)$ be another common fixed point of S^r, T^r and A^r for all $r > 0$. By inequality we write

$$M(A^r z, A^r w, a, q^r t) \geq \min\left\{\begin{matrix} M(T^r w, A^r w, a, t), \frac{M(S^r z, A^r z, a, t)}{M(A^r w, T^r w, a, t)}, \frac{M(S^r z, T^r w, a, t)}{M(S^r z, A^r w, a, t)}, \\ M(A^r z, T^r w, a, t), M(S^r z, A^r w, a, t) \end{matrix}\right\}$$

$$M(A^r z, A^r w, a, q^r t) \geq \min\left\{\begin{matrix} M(w, w, a, t), \frac{M(z, z, a, t)}{M(ww, a, t)}, \frac{M(z, w, a, t)}{M(z, w, a, t)}, \\ M(z, w, a, t), M(z, w, a, t) \end{matrix}\right\}$$

S^r and T^r $\min\{M(z, w, a, t)\}$
This implies that $M(z, w, a, q^r t) \geq \min\{M(z, w, a, t)\}$
Therefore by lemma (2.3), we get $z = w$.
Now we are generalizing above theorems in complete fuzzy 3-metric space.

Theorem 3.3

Let $(X, M, *)$ be a complete fuzzy 3-metric space and let S^r and T^r be continuous mappings of X in X, then S^r and T^r have a common fixed point in X if $\exists$ continuous mapping A^r of X into $S^r(X) \cap T^r(X)$ which commute weakly with S^r and T^r and

$$M(A^r x, A^r y, q^r t, a, b) \geq \min\left\{\begin{matrix} M(T^r y, A^r y, a, b, t), M(S^r x, A^r y, a, b, t), \\ M(S^r x, T^r y, a, b, t), M(A^r x, T^r y, a, b, t), M(S^r x, A^r y, a, b, t) \end{matrix}\right\}$$

……(3.3.1)

for all $x, y, a, b \in X, t > 0$, and $0 < q < 1$. Here $(a * b) = \min(a, b)$

And $\lim_{n\to\infty} M(x, y, z, a, b, t) = 1\ \forall\ x, y, z, a \in X$ ……… (3.3.2)

Then S^r, T^r and A^r have a unique common fixed point.

Proof

We define a sequence $\{x_n\}$ such that $A^r x_{2n} = S^r x_{2n-1}$ and $A^r x_{2n-1} = T^r x_{2n}, n = 1,2, \dots\dots$
We shall prove that $\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.3.1), we write

$$M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq \min\left\{\begin{matrix} M(T^r x_{2n+1}, A^r x_{2n+1}, a, b, q^r t), M(S^r x_{2n}, A^r x_{2n}, a, b, t), \\ M(S^r x_{2n}, T^r x_{2n+1}, a, b, t), M(A^r x_{2n}, T^r x_{2n+1}, a, b, t), \\ M(S^r x_{2n}, A^r x_{2n+1}, a, b, t) \end{matrix}\right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq \min\left\{\begin{matrix} M(A^r x_{2n}, A^r x_{2n+1}, a, b, t), M(A^r x_{2n+1}, A^r x_{2n}, a, b, t), \\ M(A^r x_{2n+1}, T^r x_{2n}, a, b, t), M(A^r x_{2n}, T^r x_{2n}, a, b, t), \\ M(A^r x_{2n+1}, A^r x_{2n+1}, a, b, t) \end{matrix}\right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq M(A^r x_{2n-1}, A^r x_{2n}, a, b, q^r t)$$

$$= min\begin{Bmatrix} M(A^r x_{2n}, A^r x_{2n+1}, a, b, t), M(A^r x_{2n+1}, A^r x_{2n}, a, b, t), \\ M(A^r x_{2n+1}, A^r x_{2n}, a, b, t), 1,1 \end{Bmatrix}$$

$$\geq min\begin{Bmatrix} M(A^r x_{2n-1}, A^r x_{2n}, a, b, t/q^r), M(A^r x_{2n}, A^r x_{2n-1}, a, b, t/q^r), \\ M(A^r x_{2n}, A^r x_{2n-1}, a, b, t/q^r), 1,1 \end{Bmatrix}$$

There fore

$M(A^r x_{2n}, A^r x_{2n+1}, a, b, q^r t) \geq M(A^r x_{2n-1}, A^r x_{2n}, a, b, t/qr$

By induction $M(A^r x_{2k}, A^r x_{2m+1}, a, b, q^r t) \geq M(A^r x_{2m}, A^r x_{2k-1}, a, b, t/q^r)$

For every $k \, and \, m \, in \, N$, Further if $2m + 1 > 2k$, then

$M(A^r x_{2k}, A^r x_{2m+1}, a, b, q^r t) \geq M\left(A^r x_{2m}, A^r x_{2k-1}, a, b, \frac{t}{q^r}\right) \geq \cdots ....$

$\ldots \ldots \ldots \ldots \ldots \geq M(A^r x_0, A^r x_{2m+1-2k}, a, b, t/q^{2kr})$ ....(3.3.3)

If $2k > 2m + 1$, then

$M(A^r x_{2k}, A^r x_{2m+1}, a, b, q^r t) \geq M\left(A^r x_{2k-1}, A^r x_{2m}, a, b, \frac{t}{q^r}\right) \geq \cdots ....$

$\ldots \ldots \ldots \ldots \ldots \geq M(A^r x_{2k-(2m+1)}, A^r x_0, a, b, t/q^{(2m+1)r})$ ....(3.3.4)

By simple induction with (3.3.3) and (3.3.4) we have

$M(A^r x_n, A^r x_{n+p}, a, b, q^r t) \geq M\left(A^r x_0, A^r x_p, a, b, \frac{t}{q^r}\right).$

For $n = 2k, p = 2m + 1 \, or \, n = 2k + 1, p = 2m + 1$ and by (FM-4)

$M(A^r x_n, A^r x_{n+p}, a, b, q^r t) \geq \left\{M\left(A^r x_0, A^r x_1, a, b, \frac{t}{2q^{nr}}\right), M\left(A^r x_1, A^r x_p, a, b, \frac{t}{q^{nr}}\right)\right\}$

...........................(3.3.5)

If $n = 2k, p = 2m \, or \, n = 2k + 1, p = 2m.$

For every positive integer $p \, and \, n \, in \, N$, by nothing that $M\left(A^r x_0, A^r x_p, a, b, \frac{t}{q^{nr}}\right) \to 1 \; as \, n \to \infty$

Thus $\{A^r x_n\}$ is a Cauchy sequence. Since the space X is complete there exists $z \in X$, such that

$\lim_{n\to\infty} A^r x_n = \lim_{n\to\infty} S^r x_{2n+1} = \lim_{n\to\infty} T^r x_{2n} = z$

It follows that $Az = Sz = Tz$ and so

$M(A^r z, A^{2r} z, a, b, qt) \geq$

$$\begin{Bmatrix} M(T^r A^r z, A^r A^r z, a, b, t), M(S^r z, A^r z, a, b, t), M(S^r z, T^r A^r z, a, b, t), \\ M(A^r z, T^r A^r z, a, b, t), M(S^r z, A^r A^r z, a, b, t), \end{Bmatrix}$$

$M(A^r z, A^{2r} z, a, b, qt) \geq M(S^r z, T^r A^r z, a, b, t)$

$\geq M(S^r z, A^r z, T^r a, b, t) \geq M(A^r z, A^{2r} z, a, b, qt) \ldots \ldots \ldots \geq M\left(A^r z, A^{2r} z, a, b, \frac{t}{q^{nr}}\right)$

$since \; \lim_{n\to\infty} M\left(A^r z, A^{2r} z, a, \frac{t}{q^{nr}}\right) = 1 \Rightarrow A^r z = A^{2r} z.$

Thus z is common fixed point of $A^r, S^r \, and \, T^r$.

For **uniqueness**, let $w \, (w \neq z)$ be another common fixed point of $S^r, T^r \, and \, A^r$. By (3.3.1) we write

$M(A^r z, A^r w, a, qt) \geq$

$$min\begin{Bmatrix} M(T^r w, A^r w, a, t), M(S^r z, A^r z, a, t), \\ M(S^r z, T^r w, a, t), M(A^r z, T^r w, a, t), M(S^r z, A^r w, a, t) \end{Bmatrix}$$

$M(A^r z, A^r w, a, qt) \geq$

$$min\begin{Bmatrix} M(w, w, a, t), M(z, z, a, t), \\ M(z, w, a, t), M(z, w, a, t), M(z, w, a, t) \end{Bmatrix}$$

$M(A^r z, A^r w, a, qt) \geq min\{M(z, w, a, t)\}$

This implies that $M(z, w, a, b, qt) \geq min\{M(z, w, a, b, t)\}$

Therefore by lemma (2.3) we get $z = w.$

This completes the proof of the theorem.

Conclusion

Above established theorems are the generalization of the earlier results of fisher [10] and all others, also we extend this result is to fuzzy 2 and 3-metric spaces by obtaining some common fixed point theorems on fuzzy metric spaces.

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